

Stabilizer information inequalities from phase space distributions

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The Shannon entropy of a collection of random variables is subject to a number of constraints, the best-known examples being monotonicity and strong subadditivity. It remains an open question to decide which of these “laws of information theory” are also respected by the von Neumann entropy of many-body quantum states. In this note, we consider a toy version of this difficult problem by analyzing the von Neumann entropy of stabilizer states. We find that the von Neumann entropy of stabilizer states satisfies all *balanced* information inequalities that hold in the classical case. Our argument relies on the fact that any stabilizer state has a classical model, provided by the discrete Wigner function: The phase-space entropy of the Wigner function corresponds directly to the von Neumann entropy of the state, which allows us to reduce to the classical case. Our result has a natural counterpart for multi-mode Gaussian states, which sheds some light on the general properties of the construction.

Remark: After the research presented in this note had been concluded, Linden, Ruskai, and Winter published a preprint which also presents an analysis of the entropy cone generated by stabilizer states [1]. While our main result is implied by [1], we believe that our approach of providing a classical model via phase space distributions is of independent interest.

I. INTRODUCTION AND RESULTS

The Shannon entropy of a discrete random variable X is given by $H(X) = -\sum_x p_x \log p_x$, where p_x is the probability that $X = x$. Given a collection of random variables X_1, \dots, X_n , we can consider the joint entropy $H(X_I)$ of any non-empty subset $X_I = (X_i)_{i \in I}$ of the variables. These entropies are not independent—they are subject to a number of linear homogeneous inequalities, known as *information inequalities*, or as the “laws of information theory” [2]. Conversely, the set of all such inequalities determines the set of possible joint entropies ($H(X_I)$) up to closure [3]. The fundamental inequalities are the *monotonicity* of the Shannon entropy, i.e. that the entropy does not decrease if more random variables are taken into account, $H(X_{I \cup J}) - H(X_I) \geq 0$, and its *strong subadditivity*,

$$H(X_I) + H(X_J) - H(X_{I \cap J}) - H(X_{I \cup J}) \geq 0.$$

These are the inequalities of *Shannon type*. Since the seminal work [4], it is known that there are entropy inequalities not implied by those of Shannon type. In fact, there are infinitely many independent such inequalities [5].

In quantum mechanics, the state of a quantum state of n particles is described by a density operator ρ on a tensor-product Hilbert space. The state of any subset $I \subseteq \{1, \dots, n\}$ of the particles is described by the reduced state $\rho_I = \text{tr}_{I^c} \rho$ formed

by tracing out the Hilbert space of the other particles. The natural analogue of the Shannon entropy is the von Neumann entropy $S(\rho) = -\text{tr} \rho \log \rho$ [6], and it is of fundamental interest to determine the linear inequalities satisfied by the entropies $S(\rho_I)$ of subsystems [7]. The most immediate difference to the classical case is that the von Neumann entropy is no longer monotonic: global quantum states can exhibit less entropy than their reductions (a signature of entanglement). Instead, it satisfies *weak monotonicity*:

$$S(\rho_{I \cup K}) + S(\rho_{J \cup K}) - S(\rho_I) - S(\rho_J) \geq 0.$$

Strong subadditivity, however, famously remains valid for quantum entropies [8]. It is a major open problem in quantum information theory to decide whether there are any entropy inequalities beyond the above (see [9, 10] for some partial progress, including a class of so-called *constrained inequalities*).

Both strong subadditivity and weak monotonicity are tight for product states (resp. for independent random variables). An entropy inequality $\sum_I \nu_I S(\rho_I) \geq 0$ has that property if and only if $\sum_{I \ni i} \nu_I = 0$ for all i . Such inequalities are called *balanced* [11] and they will play an important role below. An example of an inequality that is not balanced is given by monotonicity.

Of course, entropy inequalities are also relevant in the case of continuous variables (also classically, see e.g. [11]) as well as for other kinds of entropies, e.g. Rényi entropies [12].

In this work we study the entropy inequalities satisfied by two classes of quantum states—namely, *stabilizer states* and *Gaussian states* (which are the continuous-variable counterpart of the former). These states are versatile enough to exhibit intrinsically quantum features (such as multi-particle entanglement), but possess enough structure to allow for a concise and computationally efficient description. In both cases, *quantum phase space methods* have been built around them, and it is this point of view we aim to exploit here.

To this end, we consider the Wigner function, which for both classes of states is a bona fide probability distribution on

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a classical phase space. For n systems of dimension d , the phase space is \mathbb{Z}_d^{2n} , while for n bosonic modes it is given by \mathbb{R}^{2n} . In both cases, it is the direct sum of the single-particle/single-mode phase spaces. Given a state ρ , we then define a random variable $X = (X_1, \dots, X_n)$ on the phase space, where X_i denotes the component in the single-particle space of the i -th particle or mode. The random variables X_1, \dots, X_n constitute our classical model. This construction is compatible with reduction: the marginal probability distribution of a subset of variables $X_I = (X_i)_{i \in I}$ is given precisely by the Wigner function of the reduced quantum state ρ_I . Our crucial observation then is that certain quantum entropies are simple functions of the corresponding classical entropies. More precisely, we find that

$$S_2(\rho_I) = H_2(X_I) - C|I|, \quad (1)$$

where $C > 0$ is a universal constant and where $S_2(\rho) = -\log \text{tr} \rho^2$ and $H_2(X) = -\log \sum_x p_x^2$ denote the quantum and classical Rényi-2 entropy, respectively. In the case of continuous variables, we use the differential Rényi entropy $H_2(X) = -\log \int p_x^2 dx$. Therefore, if $\sum_I \nu_I H_2(X_I) \geq 0$ is a *balanced* entropy inequality satisfied by the random variables X_I then the same inequality is satisfied by the quantum state,

$$\sum_I \nu_I S_2(\rho_I) = \sum_I \nu_I H_2(X_I) - C \underbrace{\sum_I \nu_I |I|}_{=0} \geq 0.$$

In the case of stabilizer states (§II), all reduced states ρ_I are normalized projectors (onto the corresponding code subspace), while the X_i are uniformly distributed (on their support). Thus all Rényi entropies agree with each other, and also with the Shannon and von Neumann entropy, respectively:

$$S(\rho_I) = H(X_I) - |I|. \quad (2)$$

As above, it follows that any balanced entropy inequality that is valid for the Shannon entropies of the X_I is also valid for the von Neumann entropies of the stabilizer states ρ_I . In particular, *stabilizer states respect all balanced information inequalities*, such as the inequalities of non-Shannon type found in [13]. What is more, our construction can also be understood in the group-theoretical framework of [14]. Here it is well-known that there are inequalities which do not hold for arbitrary random variables, but only for random variables constructed from certain classes of subgroups, e.g. normal subgroups [15]. Since phase spaces are Abelian groups, it follows that the von Neumann entropy of stabilizer states also respect such information laws, e.g. the *Ingleton inequality* [15], which is the balanced inequality

$$I_\rho(I : J|K) + I_\rho(I : J|L) + I_\rho(K : L) - I_\rho(I : J) \geq 0. \quad (3)$$

Here, $I_\rho(I : J) = S(\rho_I) + S(\rho_J) - S(\rho_{I \cup J})$ and $I_\rho(I : J|K) = S(\rho_{I \cup K}) + S(\rho_{J \cup K}) - S(\rho_K) - S(\rho_{IJK})$ are the quantum (conditional) mutual information.

We find it instructive to understand how the above classical model manages to respect monotonicity, while the quantum

state may violate it. For example, since stabilizer states can be entangled (even maximally so), $H(\rho_1) = H(\rho_2) = 1$ and $H(\rho_{12}) = 0$ are perfectly valid entropies of a stabilizer state which obviously violate monotonicity. Equation (2) states that the classical model is *more highly mixed* than the quantum one, in the sense that the entropy associated with a subset I is higher by an amount of $|I|$. That is precisely the amount by which quantum mechanics can violate monotonicity.

In the case of Gaussian states (§III), the random variables X_1, \dots, X_n have a multivariate normal distribution, and we show that the differential Rényi-2 entropy in (1) can be replaced by the Rényi- α entropy for arbitrary positive $\alpha \neq 1$:

$$S_2(\rho_I) = H_\alpha(X_I) - |I| \left(\log \pi - \frac{\log \alpha}{1 - \alpha} \right),$$

In the limiting case $\alpha \rightarrow 1$, we recover a formula involving the differential Shannon entropy which has previously appeared in [16], attributed to Stratonovich. Thus, *Rényi-2 entropies of Gaussian states respect all balanced information inequalities* that hold for multivariate normal distributions (see [17, 18]). Interestingly, it is not clear whether a similar statement holds for the von Neumann entropy of the Gaussian state. This is perhaps an indication that the connection (2) between the Shannon and the von Neumann entropy for stabilizer states is somewhat coincidental. The comparison with Gaussian states suggests that the Rényi-2 entropies might be the more fundamental quantities in this context, that merely happen to agree with the von Neumann entropy in the case of stabilizer states.

We conclude this section with a few remarks. While it is known that the Wigner function approach cannot be straight-forwardly translated to non-stabilizer states [19–21], our discussion suggests searching for other maps from quantum states to probability distributions that reproduce entropies faithfully, up to state-independent additive constants.

Our note uses the classical model provided by the Wigner function as a tool for proving statements that do not, a priori, seem to be connected to phase space distributions. This point of view has been employed before, e.g. to construct quantum expanders [22], to establish simulation algorithms [23–25], and for demonstrating the onset of contextuality [26]. It would be interesting to see further applications.

In order to establish the Ingleton inequality (3), we have invoked the group-theoretical approach to classical information inequalities due to [14]. It would be highly desirable to find a quantum-mechanical analogue of this work (see [27, 28] for partial results towards this goal, motivated by the quantum marginal problem of quantum physics).

Convention. In this work, entropies of d -level systems are measured in units of $\log d$ bits. In the continuous-variable case, we employ the natural logarithm.

II. STABILIZER STATES

In this section, we provide the technical details of our results on stabilizer states. The presentation is slightly complicated by the fact that the phase space methods we employ

work somewhat less well in even dimensions than in odd ones. The status is as follows: Phase spaces and stabilizer states can be defined for any dimension d . Indeed, our main result is stated and proved for any d . If the dimension is odd, there is an additional piece of structure – namely a discrete Wigner function that replicates many properties of its better-known continuous-variable variant [20]. For odd d , one obtains the classical model that we construct simply by interpreting the non-negative discrete Wigner function of a stabilizer code as a probability distribution. This observation was the original motivation for our general construction.

We start by fixing some notation and recall the basic formalism of stabilizer states [6, 29]. Our presentation is centered on the phase space picture, see [20] for a more detailed discussion.

The discrete *configuration space* of a single particle is \mathbb{Z}_d , where \mathbb{Z}_d is the set $\{0, \dots, d-1\}$ with arithmetic modulo d . The associated *phase space* is $\mathbb{Z}_d^2 = \mathbb{Z}_d \oplus \mathbb{Z}_d$. We denote the components of vectors $v \in \mathbb{Z}_d^2$ by (p, q) in order to emphasize the analogy with momentum and position in the continuous-variable theory. A crucial piece of structure is the *symplectic form* defined on the phase space. It maps vectors $v = (p, q)$ and $v' = (p', q')$ to $[v, v'] = pq' - qp'$. A minor complication in even dimensions is that the form must there be understood as taking values in \mathbb{Z}_{2d} , i.e. the arithmetic in $pq' - qp'$ is to be taken modulo $2d$ [30, 31]. There is a representation of \mathbb{Z}_d^2 in terms of Weyl operators acting on the Hilbert space of complex functions on \mathbb{Z}_d , which we identify with \mathbb{C}^d . It is given by $(w(p, q)\psi)(x) = e^{i\frac{\pi}{d}(2px - pq)}\psi(x - q)$. A straightforward calculation shows that

$$w(v)w(v') = e^{i\frac{\pi}{d}[v, v']}w(v + v'). \quad (4)$$

Thus the Weyl operators realize a *projective* or *twisted* representation of the additive group of the phase space (it is a faithful representation of the *Heisenberg group* over \mathbb{Z}_d [32]).

For n particles, the phase space is the direct sum $V = \bigoplus_{i=1}^n V_i = \mathbb{Z}_d^{2n}$ of the single-particle phase spaces $V_i = \mathbb{Z}_d^2$. It can be represented on $(\mathbb{C}^d)^{\otimes n}$ by the tensor product of the single-particle representations, $w(v) = \bigotimes_{i=1}^n w(v_i)$, and the composition law (4) continuous to hold if we extend the symplectic form linearly.

Let us now consider an *isotropic subspace* $M \subseteq V$, i.e. a subspace on which the symplectic inner product vanishes. Assume for a moment that d is odd. Isotropicity implies by Eq. (4) that the Weyl operators $\{w(m) \mid m \in M\}$ all commute. Using this fact, one easily verifies that

$$P(M) := \frac{1}{|M|} \sum_{m \in M} w(m) \quad (5)$$

defines an orthogonal projection onto a $d^{n-\dim M}$ -dimensional subspace of $(\mathbb{C}^d)^{\otimes n}$. This subspace is called the *stabilizer code* associated with M . The corresponding *stabilizer state* then is the normalized projection

$$\rho(M) = \frac{1}{\text{tr } P(M)} P(M).$$

(One obtains a larger set of stabilizer codes by including certain phase factors in the sum in Eq. (5) [6, 20, 29]. However, all stabilizer codes are locally equivalent to one of the form (5), so we do not incur a loss of generality). For even dimensions, the construction is again slightly more complicated. Here, one has to choose a basis $\mathcal{B} = \{m_1, \dots, m_k\}$ of M (we have set $k := \dim M$) and define

$$P(\mathcal{B}) := \frac{1}{|M|} \sum_{x \in \mathbb{Z}_d^n} w(m_1)^{x_1} \dots w(m_k)^{x_k}.$$

One can then repeat the previous calculation to find that $P(\mathcal{B})$ is again a $d^{n-\dim M}$ -dimensional projection. We record that in both cases the von Neumann entropy of a stabilizer state is given by

$$S(\rho(M)) = n - \dim M. \quad (6)$$

Using the basic relation $\text{tr } w(v) = d^n \delta_{v,0}$, one obtains a simple expression for the reduced state $\rho(M)_I$. For this, let $V_I := \{v \in V : v_i = 0 \text{ for } i \notin I\}$ be the phase space of some subset of particles $I \subseteq \{1, \dots, n\}$, and set $M_I := M \cap V_I$. Then

$$\rho(M)_I = \rho(M_I), \quad (7)$$

i.e. the reduced state is the stabilizer state described by the isotropic subspace $M_I \subseteq V_I$. From (6) we find that

$$S(\rho(M)_I) = S(\rho(M_I)) = |I| - \dim M_I. \quad (8)$$

We now describe the construction of the classical model for odd dimensions. To this end, let us consider the (*discrete*) *Wigner function* of a quantum state ρ , which is defined to be the function on phase space given by

$$W_\rho(v) = d^{-2n} \sum_{w \in V} e^{-i\frac{\pi}{d}[v, w]} \text{tr } (w(w)^\dagger \rho).$$

This definition is in analogy to the continuous-variable case, as explained in detail in [20]. The central observation is that in the case of stabilizer states, the Wigner function $W_{\rho(M)}$ formally defines a *probability distribution on phase space*, i.e. it attains only non-negative values and their sum is one. In fact, we can easily compute (c.f. [20, 33])

$$\begin{aligned} W_{\rho(M)}(a) &= d^{-2n} \sum_{b \in V} e^{-i\frac{\pi}{d}[a, b]} \delta_M(b) \\ &= d^{-2n+\dim M} \delta_{M^\perp}(a) \\ &= \frac{1}{|M^\perp|} \delta_{M^\perp}(a), \end{aligned} \quad (9)$$

where we have defined the *symplectic complement* by $M^\perp = \{v \in V : [v, m] = 0 \ \forall m \in M\}$. Therefore, the Wigner function of a stabilizer state with isotropic subspace $M \subseteq V$ is given precisely by the uniform distribution on $M^\perp \subseteq V$.

Our main observation now is that this construction defines a classical model for any given stabilizer state which reproduces the entropies of all reduced states up to a certain constant. We phrase this result without recourse to Wigner functions in such a way that it holds for arbitrary local dimensions, even or odd.

Theorem 1. Let $V = \bigoplus_{i=1}^n V_i = \mathbb{Z}_d^{2n}$ be the phase space for n particles with local dimension d , where $d > 1$ is an arbitrary integer. Let $\rho(M)$ be the stabilizer state corresponding to an isotropic subspace $M \subseteq V$, and define a random variable $X = (X_1, \dots, X_n)$ that takes values uniformly in the symplectic complement $M^\perp \subseteq V$. Then,

$$S(\rho(M)_I) = H(X_I) - |I|, \quad (10)$$

and the same conclusion holds if we replace the Shannon and von Neumann entropy by any Rényi entropy.

If d is odd then the above construction can be obtained by interpreting the Wigner function $W_{\rho(M)}$ as the probability distribution of the random variable X .

Proof. To prove (10), denote by $\pi_I: V \rightarrow V_I$ the projection onto the phase space of parties $I \subseteq \{1, \dots, n\}$. It will be convenient to consider V_I as a subspace of V in the natural way. To avoid any notational ambiguity, we denote by X^{\perp_I} the symplectic complement of a subspace X taken within V_I .

Observe that

$$\pi_I(M^\perp) \subseteq M_I^{\perp_I}.$$

Indeed, if $v \in M^\perp$ and $m_I \in M_I$, then $[\pi_I(v), m_I] = [v, m_I] = 0$. On the other hand, we find that

$$\pi_I(M^\perp)^{\perp_I} \subseteq M_I$$

To see this, consider a vector $v_I \in V_I$ and note that if $v_I \perp \pi_I(M^\perp)$ then $v_I \perp M^\perp$, hence $v_I \in M \cap V_I = M_I$. We conclude that

$$\pi_I(M^\perp) = M_I^{\perp_I}. \quad (11)$$

Note that $X_I = \pi_I(X)$. Since π_I is a group homomorphism, it follows that X_I is distributed uniformly on its range, so that

$$\begin{aligned} H(X_I) &= \log|\pi_I(M^\perp)| = \log|M_I^{\perp_I}| = \dim M_I^{\perp_I} \\ &= 2|I| - \dim M_I = |I| + S(\rho(M)_I), \end{aligned}$$

where we have used (8) in the last step. We have thus established (10).

The same result holds if we replace the Shannon and von Neumann entropy by Rényi entropies. This is because the stabilizer states $\rho(M)_I$ are normalized projectors and each random variable X_I is distributed uniformly on its range, so that the entropies coincide.

Finally, it is clear from (9) that for odd d the distribution of X coincides with the Wigner function $W_{\rho(M)}$ of the stabilizer state. It remains to show that the Wigner function W_{ρ_I} of a reduced state ρ_I is obtained by marginalizing the full Wigner function (in other words: the quantum and the classical way of reducing to subsystems commute):

$$W_{\rho_I}(v) = \sum_{w: w_I = v} W_\rho(w) \quad (12)$$

for all $v \in V_I$. While this can easily be proved in full generality from the definition of the Wigner function, it is also true that for the special case of stabilizer states, Eq. (12) follows directly from (11). \square

Corollary 2. Stabilizer states satisfy all balanced information inequalities. Moreover, they satisfy the Ingleton inequality (3).

Proof. As described in the introduction, the first claim follows immediately from (10). This is because for any balanced information inequality $\sum_I \nu_I H(X_I) \geq 0$ we necessarily have that [18]

$$\sum_I \nu_I |I| = \sum_I \left(\sum_{i \in I} \nu_I \right) = \sum_i \left(\sum_{I \ni i} \nu_I \right) = 0.$$

Hence the correction term in (10) cancels as we sum over all subsystems:

$$\sum_I \nu_I S(\rho(M)_I) = \sum_I \nu_I H(X_I) - \sum_I \nu_I |I| \geq 0.$$

For the second claim, observe that the random variables X_I are given as the image of a randomly chosen vector $X \in M^\perp$ along the linear projection $\pi_I: V \rightarrow V_I$. Thus a corollary in [15] shows that the Ingleton inequality (3) is satisfied (since it is evidently balanced). In the language of [14, 15], this is because the entropy vector $(H(X_I))$ can be characterized by normal subgroups (in fact, our phase spaces are even Abelian groups). \square

III. GAUSSIAN STATES

We sketch the corresponding result for Gaussian states of continuous-variable systems. The Wigner function of an n -mode Gaussian quantum state ρ with covariance matrix Σ and first moments μ is defined as follows on classical phase space \mathbb{R}^{2n} :

$$W_\rho(x) = \frac{1}{(2\pi)^n (\det \Sigma)^{\frac{1}{2}}} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)},$$

(see e.g. the review [34]). Evidently, W_ρ is the probability density of a random vector $X = (X_1, \dots, X_{2n})$ with multivariate normal distribution of mean μ and covariance matrix Σ . Using the well-known relation $\text{tr} \rho^2 = (2\pi)^n \int W_\rho^2(x) dx$, it follows that the Rényi-2 entropy of the quantum state, $S_2(\rho) = -\log \text{tr} \rho^2$, is directly related to the differential Rényi-2 entropy of the random variable X , $H_2(X) = -\log \int W_\rho^2(x) dx$:

$$S_2(\rho) = H_2(X) - n \log(2\pi). \quad (13)$$

The reduced state ρ_I for some subset of modes $I \subseteq \{1, \dots, n\}$ is again a Gaussian state, and its covariance matrix is equal to the corresponding submatrix of Σ . Thus the Wigner function of ρ_I is given by the marginal probability density of the variables $X_I = (X_i)_{i \in I}$, and using (13) we find that

$$S_2(\rho_I) = H_2(X_I) - |I| \log(2\pi). \quad (14)$$

Equation (14) states that the Rényi-2 entropy of a Gaussian quantum state is always lower than the phase space entropy of its classical model, as given by the Wigner function. It is so by a precise amount, namely by $\log(2\pi)$ bits per mode.

Theorem 3. Let ρ be a Gaussian state with covariance matrix Σ , and define a random variable $X = (X_1, \dots, X_n)$ with probability density given by the Wigner function $W_\rho(x)$. Then, for any positive $\alpha \neq 1$,

$$S_2(\rho_I) = H_\alpha(X_I) - |I| \left(\log \pi - \frac{\log \alpha}{1 - \alpha} \right),$$

where $H_\alpha(X) = 1/(1 - \alpha) \log \int W_\rho^\alpha(x) dx$ is the differential Rényi- α entropy. In the limit $\alpha \rightarrow 1$, we recover

$$S_2(\rho_I) = H(X_I) - |I| (\log \pi + 1). \quad (15)$$

where $H(X) = - \int W_\rho(x) \log W_\rho(x) dx$ is the differential Shannon entropy.

Proof. It is an easy exercise in Gaussian integration to show that the differential Rényi- α entropy of the random variable X_I is given by

$$H_\alpha(X_I) = \frac{1}{2} \log \det \Sigma + |I| \left(\log 2\pi - \frac{\log \alpha}{1 - \alpha} \right).$$

The assertions of the theorem follow from this and (14). \square

Equation (15) has been previously used in [16], where the formula is attributed to Stratonovich. Just as in the discrete case, we immediately get the following corollary:

Corollary 4. The Rényi-2 entropy for Gaussian states satisfies all balanced information inequalities that are valid for multivariate normal distributions.

Interestingly, Gaussian states can violate the Ingleton inequality (as opposed to stabilizer states, cf. Corollary 2). Indeed, this is well-known for multivariate normal distributions, and it is readily verified that the counterexample presented in [18] is a physical covariance matrix of a Gaussian state (i.e., it satisfies the *uncertainty relation* $\Sigma + i\Omega \geq 0$, where Ω is the symplectic matrix). The claim thus follows from Theorem 3.

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